8. STELLAR DYNAMICS II : 3-D SYSTEMS

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(1) Introduction

We have, of course, already begun our study of Stellar Dynamics :
Topic 6 considered the highly restricted situation of nearly circular motion in cool galaxy disks.
Here we broaden the discussion considerably to consider motion within more general 3-D systems.
In large part, these notes follow (though simplify) the treatment in B&T.

(a) Gas/Fluid Physics and Stellar Dynamics

To set the stage, lets first compare **stellar systems** with atomic (or molecular) **gases** :

- First, some **similarities** :
  Each comprise **many**, **interacting** objects which act as **points** (separation >> size)
  Each can be described by **distributions** in **space** and **velocity**
  eg Maxwellian velocity distributions; uniform density; spherically concentrated etc.
  Stars or atoms are neither created nor destroyed -- they both obey **continuity equations**
  All interactions as well as the system as a whole obeys **conservation laws** (eg energy, momentum)

- Now some crucial **differences** :
  The relative importance of short and long range forces is radically different :
  -- atoms interact only with their **neighbors**, during brief elastic repulsive collisions
  -- stars interact continuously with the **entire ensemble** via the long range attractive force of gravity
  eg uniform medium : $F \propto G \rho \Omega dr / r^2 \propto \rho r^2 dr / r^2 \propto \rho dr \rightarrow$ equal force from all distances
  The relative frequency of strong encounters is radiaclly different :
  -- for atoms, encounters are **frequent** and all are **strong** (ie $\Delta V \sim V$)
  -- for stars, pairwise encounters are **very rare**, and the stars move in the smooth global potential

- Consequently, there are many **parallels** between gas (fluid) dynamics and stellar dynamics :
  ---> concepts such as Temperature and Pressure can be applied to stellar systems
  ---> we use analogs to the equations of fluid dynamics and hydrostatics

- there are also some interesting **differences**
  ---> pressures in stellar systmes can be **anisotropic**
  ---> stellar systems have **negative** specific heat and evolve **away** from uniform temperature.

(b) A Path Through the Subject

There are a number of themes to cover, and chosing the right sequence isn't straightforward

Here is an outline to help navigate the upcoming (sometimes dense) material.

- The geometry of **gravitational potentials** is a good starting point :
  → methods to derive gravitational potentials from mass distributions, and visa versa.

- Potentials define how stars move
  → consider stellar **orbit shapes**, and divide them into **orbit classes**.

- The gravitational field and stellar motion are deeply interconnected :
  → the **Virial Theorem** relates the global potential energy and kinetic energy of the system.
  The Virial Theorem can be used to investigate :
  → the masses of stellar systems
  → how energy is released during gravitational collapse
  → how self-gravitating systems have negative specific heat.
  → how the ratio of rotation to dispersion support can define galaxy flattening.

- A more detailed approach requires us to work with a **Distribution Function** (DF) :
  → the DF specifies how stars are distributed throughout the system and with what velocities.

- For **collisionless systems**, the DF is constrained by a continuity equation : the **CBE**
  This can be recast in more observational terms as the **Jeans Equation**.
  The **Jeans Theorem** helps us choose DFs which are solutions to the continuity equations.

- With these DFs, we can construct **self-consistent models** of **equilibrium** stellar systems.
  Some simple systems are considered in detail, while more complex ones are touched on.

- We introduce situations where the potential is **changing in time**
  Usually this is untreatable, except when the changes are rapid and large : **violent relaxation**
  This is important in describing galaxy formation and galaxy merging

- Finally, we relax the collisionless assumption and introduce **star-star interactions**
  → such systems are described by the **Fokker-Planck** equation
  This reveals a number of slow processes which occur in dense stellar systems :
  → 2-Body relaxation & equipartition
  → Core collapse & the gravothermal catastrophe
  → Evaportion & ejection

- Additional important themes are postponed to later Topics :
  → Effect of nuclear black holes on stellar distributions (9)
  → Dynamical friction (15)
  → Tidal evaporation (15)
  → Slow (adiabatic) and Fast (impulsive) encounters (15)
  → Merging & satellite accretion (15)

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(2) Potential Theory

(a) Preliminaries

- We initially characterize mass distributions as **smooth** functions ρ(r)
  (this is usually legitimate for galaxies, see § 8.10 below)

- The gravitational **potential energy** is a **scalar field**
  its gradient gives the net gravitational **force** (per unit mass) which is a **vector field** :

$$\Phi(\mathbf{r}) = -G \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|}\, d^3\mathbf{r}' \tag{8.1a}$$

$$\mathbf{F}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) = G \int_V \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3}\, \rho(\mathbf{r}')\, d^3\mathbf{r}' \tag{8.1b}$$

- evaluating the divergence of **F**(**r**) gives :

$$\nabla \cdot \mathbf{F}(\mathbf{r}) = -4\pi G \rho(\mathbf{r}) \quad (8.2a)$$

$$\nabla^2 \Phi(\mathbf{r}) = 4\pi G \rho(\mathbf{r}) \quad (8.2b)$$

$$\nabla^2 \Phi(\mathbf{r}) = 0 \quad (8.2c)$$

8.2b is Poisson's equation, for locations **within** the mass distribution
8.2c is Laplace's equation, for locations **outside** the mass distribution

- For a volume V with surface A enclosing mass M we have (using Divergence/Gauss's Theorem) :

$$4\pi GM = 4\pi G \int_V \rho(\mathbf{r})\, d^3\mathbf{r} \quad (8.3a)$$

$$= \int_V -\nabla \cdot \mathbf{F}(\mathbf{r})\, d^3\mathbf{r} = \int_A -\mathbf{F}(\mathbf{r}) \cdot d^2\mathbf{S} \quad (8.3b)$$

- Since the force field is the gradient of a potential, it is **conservative**,
ie the energy required to move mass from $\mathbf{r}_1$ to $\mathbf{r}_2$ is **independent** of the path
the total **Potential Energy** is therefore **well defined**
setting $\Phi = 0$ at $r = \infty$ we get (B&T-2 p 59) :

$$W = \frac{1}{2}\int_V \rho(\mathbf{r})\,\Phi(\mathbf{r})\, d^3\mathbf{r} = -\frac{1}{8\pi G}\int_V |\nabla\Phi|^2\, d^3\mathbf{r} \quad (8.4)$$

Note that, with this definition, potential energy is **always negative**

(b) Selected Examples of Density-Potential Pairs

Often, choosing a simple form for $\rho(\mathbf{r})$ [or $\Phi(\mathbf{r})$] yields a complex form for $\Phi(\mathbf{r})$ [or $\rho(\mathbf{r})$]
There are, however, a number of useful illustrative analytic $\rho(\mathbf{r}) \leftrightarrow \Phi(\mathbf{r})$ pairs :

(i) Point Mass

$\Phi(r) = -GM / r$; $F(r) = -\nabla \Phi = -d\Phi/dr = -GM / r^2$
$V_c^2(r) = GM / r = -\Phi(r)$; $V_{esc}^2(r) = 2GM / r = -2\Phi(r)$

where V_c & V_{esc} are the circular and escape velocities, respectively.

This is called a **Keplerian Potential**, since it pertains to the solar system.

(ii) Uniform Spherical Shell

Outside : $\Phi(r) = -GM / r$ (Keplerian)
Inside : $\Phi(r) = \text{const}$; $F(r) = 0$

(iii) Homogeneous Sphere

Sphere radius = a, with $\rho(r) = \text{const}$ $(r < a)$
Outside : $\Phi(r) = -GM / r$ (Keplerian)
Inside : $\Phi(r) = -2\pi G\rho(a^2 - r^2/3)$; $F_r = -G\,M(r) / r^2 = -(4/3)\pi G \rho \times r$

which gives SHM with period $P_r = (3\pi / G\rho)^{1/2}$ and free-fall $t_{ff} \sim 1/4\, P_r \sim (G\rho)^{-1/2}$

$V_c = [(4/3)\pi G\rho]^{1/2} \times r$ so that $\Omega(r) = \text{const}$ $\rightarrow$ **solid body rotation**

note also that $P_c = P_r$

(iv) Logarithmic Potentials from Flat Rotation Curves

Many rotation curves are **flat** at large radii : $V_c = V_o$, so we have :

$$\frac{V_0^2}{r} = F_r = -\frac{d\Phi}{dr} \quad ; \quad \Phi(r) = V_0^2 \ln r + const \tag{8.5}$$

(v) Spherical Systems

- Power Laws : $\rho = \rho_o (r/a)^{-\alpha}$
  have $M(<r) = (4 \pi G a^3 \rho_o) / (3 - \alpha) \times (r / a)^{3-\alpha}$
  and $\Phi(r) = -(4 \pi G a^2 \rho_o) / [(3 - \alpha)(\alpha - 2)] \times (r / a)^{2-\alpha} = V_c^2 / (\alpha - 2)$
  $\alpha = 3$ is a break point:
  For $\alpha > 3$, $M(<r) \to \infty$ for $r \to 0$: we have infinite mass at the origin.
  For $\alpha < 3$, $M(<r) \to \infty$ for $r \to \infty$: mass diverges at large r.
  However for $2 < \alpha < 3$ the potential is finite, as are V_c and V_{esc}, at all radii.

  The case $\alpha = 2$ is special : it is the **singular isothermal sphere**
  with $V_c = (4 \pi G a^2 \rho_o)^{1/2}$ = const at **all** radii, yielding $\Phi(r) = 4 \pi G a^2 \rho_o \ln(r / a)$
  See § 8.8a,b for other isothermal and related (King) spheres [link]

- Hernquist (1990) and Jaffe (1983) models : have $\rho \propto r^{-4}$ at large r
  which fits E gals well, and is theoretically grounded in violent relaxation
  at small r, Jaffe core is steeper than Hernquist core :

$$\rho_H(r) = \frac{Ma}{2\pi r(r+a)^3} \quad ; \quad \Phi_H(r) = -\frac{GM}{(r+a)} \tag{8.6a}$$

$$\rho_J(r) = \frac{Ma}{4\pi r^2(r+a)^2} \quad ; \quad \Phi_J(r) = \frac{GM}{a} \ln\left(\frac{r}{r+a}\right) \tag{8.6b}$$

- Plummer (1911) Sphere : is analytic solution of hydrostatic support for
  polytropic stellar system of index 5; see § 8.8c : [link]
  $\rho(r)$ matches GCs well, but is too steep at large r for Ellipticals ($\rho \propto r^{-5}$).

$$\rho_P(r) = \left(\frac{3M}{4\pi b^3}\right)\left(1 + \frac{r^2}{b^2}\right)^{-5/2} \quad ; \quad \Phi_P(r) = -\frac{GM}{\sqrt{r^2+b^2}} \tag{8.7}$$

- Plummer; Isothermal; Jaffe; and Hernquist density laws are shown here : [image]

(vi) Axisymmetric Thin Disks

- Before considering global potentials for disks, first consider the **vertical** potential near z = 0
  We have two conditions :
  within a disk of volume density ρ_o near the plane
  above a disk of surface density Σ
  Using equation 8.3b we have :

$$-\frac{\partial \Phi}{\partial z} = g_z = 4\pi G \rho_0 \, z \quad \text{(inside)} \tag{8.8a}$$

$$= 2\pi G \Sigma \quad \text{(above)} \tag{8.8b}$$

- Usually, calculating global Φ and **F** for disks is algebraically dense.
  Unlike spherical systems, disk potentials usually depend on mass **outside** R.

  Here are two examples :

- Mestel's disk : $\Sigma(R) = \Sigma_o R_o / R$, has **constant** V_c : $V_c^2(R) = 2\pi G \Sigma_o R_o = GM(<R) / R$
  this is **unusual** in that $V_c(R)$ **doesn't** depend on mass outside R

- Exponential disk : $\Sigma(R) = \Sigma_o \exp(-R/R_d)$
  this fits the light profile of sprial disks much better than Mestel's disk, and has circular velocity

$$V_c^2(R) = 4\pi G \Sigma_0 R_d y^2 [I_0(y)K_0(y) - I_1(y)K_1(y)] \quad (8.9)$$

where $y = R / 2R_d$, and I_n K_n are Bessel functions or the 1st and 2nd kind
see [Topic 5.6a] for an analytic approximation and rotation curve.

(vii) Axisymmetric Flattened Systems

Spirals with bulge and disk are, of course, neither just spherical nor just thin disks
We need potentials which are both combined, ie **flattened potentials**

- Miyamoto-Nagai (1975) flattened system [images]:
  reduces to the Plummer model if a=0 and the Kuzmin disk if b=0
  (Satoh flattened systems are derived in similar manner to the Toomre disks) [images] :

$$\rho_M(R,z) = \left(\frac{Mb^2}{4\pi}\right) \frac{aR^2 + (a+3B)(a+B)^2}{[R^2 + (a+B)^2]^{5/2} B^3} \quad (8.11a)$$

$$\Phi_M(R,z) = -\frac{GM}{\sqrt{R^2 + (a+B)^2}} \quad ; \quad B^2 = z^2 + b^2 \quad (8.11b)$$

$$\rho_{S_n}(R,z) = \left(\frac{d}{db^2}\right)^n \rho_M \quad ; \quad \Phi_{S_n}(R,z) = \left(\frac{d}{db^2}\right)^n \Phi_M \quad (8.11c)$$

(viii) Triaxial Ellipsoids

- more complicated, (see B&T-2 § 2.5)

(ix) Multipole Expansion

An **arbitrary** mass distribution ≡ sums of spherical shells of non-uniform surface density.
Calculating the potential involves solving $\nabla^2\phi = 0$ in **spherical polar** coordinates
Solutions involve **spherical harmonics** : $Y_{l,m}(\theta, \phi) \propto P_l^{|m|}(\cos\theta) \exp(i\, m\, \phi)$
where $P_l^{|m|}(x)$ are associated Legendre functions.
The potential $\Phi(r,\theta,\phi)$ is the sum of a monopole (l=0), a dipole (l=2) quadrupole (l=4) etc...
each with associated amplitudes

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(3) Orbit Classes

TBD

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(4) Numerical N-Body Methods

- Often, astrophysically interesting systems are algebraically intractable
  Computational methods provide a way forward

- Ironically, employing the Newtonian force law **can** be a disaster
  "**hard**" force law $\propto (\Delta r)^{-2}$
  close encounters give big accelarations and require **small** timesteps to follow
  if a tight binary forms, this can be a computational sink

- so "**soften**" the force law $\propto (\Delta r / (\Delta r^2 + \epsilon^2)^{-3/2}$
  (note : this may be inappropriate for small systems where "collisions" are important)

Several methods are used :
See B&T-2 § 2.9 and Josh Barnes's nice writeup for more details : [link]

- Direct Summation of pairwise forces
  only posible for N < 50000 ; ~ $O(N^2)$ operations per timestep

- Divide region into cartesian cells : population in each cell changes
  $(\Delta r)^{-2}$ only evaluated **once**
  summing done using FFTs since $\Phi_i = \sum M_j\, G\, (\Delta r_{ij})^{-2}$ (j=1,N) resembles a convolution
  takes $(2N)^2[1 + 4\log_2(2N)\,]^2$ steps compared to N^4 so very efficient for N > 16.
  Typically, 32 × 32 × 32 cube (32768 cells) with 10^5 stars
  doesn't work well for strong density gradients (eg E's) or galaxy collisions (many empty cells)
  for centrally concentrated disks, choose polar grid spaced in Ln(R) and Φ

- Express potential of mass at r,θ,ϕ by series of spherical harmonics (l < 4 often sufficient)
  calculate total potential by summing these over all particles
  resolution naturally better near nucleus
  ~ N calculations per timestep so very efficient.

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(5) The Virial Theorem

This fundamental result describes how the total energy (E) of a self-gravitating system is shared between kinetic energy (K) and potential energy (W)
Specifically, we are interested in their **ratio** : ξ = K / |W| (note K is always +ve, W always -ve)

We begin by looking at two illustrative cases and then deal with the general case.

(a) Simple Illustrations

(i) Circular Orbit

- Consider a satellite mass m in circular orbit about M (>>m) : $m V^2 / r = G m M / r^2$
  multiply by r : $m V^2 = G m M / r$ → 2K = -W or 2K + W = 0
  → ξ = K / |W| = ½ and E = - K
  → Kinetic energy is half the (-ve) potential energy
  → The total energy E = K + W is -ve and equal to (minus) the kinetic energy

- As we shall see, ξ = ½ is a characteristic shared by a wide range of systems.
  Note that in this case, the instantaneous values are also equal to the time averaged values

(ii) Time Averaged Keplerian Orbit

- In general, ξ = K / |W| **changes** along a Keplerian orbit path [image].
  eg compare ξ at pericenter and apocenter :
  $\xi_p / \xi_a = r_a / r_p \neq 1$ (using $r_p V_p = r_a V_a$ from AM conservation)

- However, taking time averages over an orbit, we find :
  $< -W > = < GM/r > = GM < 1/r > = GM \times (1/a)$, and
  $< K > = < ½ V^2 > = GM < (1/r - 1/2a) > = ½GM \times (1/a)$
  → and we recover, once again : $<\xi>$ = ½ and E = -< K >

- Note that time averages for single non-Keplerian orbits do **not** usually have $<\xi>$ = ½
  As we will see, however, ξ = ½ **always** holds when we average over **all** particles in a system
  For our Keplerian orbit, m and M **are** the whole system (with M having ~zero KE)

(b) The General Case

The general case comprises an **isolated** system of **self-gravitating** masses (see pdf)
Once again, we ask what is ξ, the ratio of kinetic to potential energies

- There are 3 equations of motion for member α (i represents x, y, z) :

$$\frac{d}{dt}(m^\alpha v_i^\alpha) \;=\; F_i^\alpha \;=\; -Gm^\alpha \sum_{\beta\neq\alpha} m^\beta \frac{r_i^\alpha - r_i^\beta}{|\mathbf{r}^\alpha - \mathbf{r}^\beta|^3} \tag{8.12}$$

- take the 1st moment in position : multiply by r_j^α and sum over α (j represents x,y,z)
  dimensionally, we have changed an equation of **forces** into an equation of **energies**
  after some algebra, we get a set of 9 equations
  these can be neatly written using 3 × 3 matrices (i.e. tensors of order 2)
  this set of equations constitute the **Tensor Virial Theorem** :

$$\boxed{\frac{1}{2}\frac{d^2}{dt^2}I_{i,j} = 2\,K_{i,j} + W_{i,j} = 2\,T_{i,j} + \Pi_{i,j} + W_{i,j}} \qquad (8.13)$$

where the five tensors are :

$$\begin{aligned}
I_{i,j} &= \int \rho\, r_i\, r_j\, d^3r &&= \text{moment of inertia} \\
K_{i,j} &= \int \tfrac{1}{2}\,\rho\,\langle v_i v_j\rangle\, d^3r &&= \text{total KE} \\
T_{i,j} &= \int \tfrac{1}{2}\,\rho\,\langle v_i\rangle\langle v_j\rangle\, d^3r &&= \text{ordered KE} \\
\Pi_{i,j} &= \int \rho\, \sigma_{i,j}^2\, d^3r &&= \text{dispersion KE} \\
W_{i,j} &= -\frac{1}{2}G \int\!\!\int \rho(\mathbf{r})\,\rho(\mathbf{r}')\,\frac{(r_i - r_i')\,(r_j - r_j')}{|\mathbf{r}' - \mathbf{r}|^3}\, d^3r\, d^3r' &&= PE
\end{aligned} \qquad \begin{matrix}(8.14)\\ (a,b,c,d,e)\end{matrix}$$

where $\sigma_{i,j}$ arises from the expansion: $\langle v_i\, v_j\rangle = \langle v_i\rangle\langle v_j\rangle + \sigma_{i,j}^2$

- For **steady state** systems, $d^2I_{ij} / d\, t^2 = 0$ and we get

$$\boxed{2K_{ij} + W_{ij} = 0} \qquad (8.15a)$$

  the Kinetic and potential energies are related **for each tensor element**
  for example, they are related separately along each axis

- Considering just the diagonal terms, we also have :
  Trace(**T**) + ½ Trace(Π) ≡ K = total kinetic energy, and
  Trace(**W**) ≡ W = total potential energy
  so for the static case, we get the **Scalar Virial Theorem** :

$$\boxed{2K + W = 0} \qquad (8.15b)$$

- Considering the **total** energy, E, we find :

$$\boxed{E = K + W = -K = \tfrac{1}{2}W} \qquad (8.15c)$$

  So the **total** energy is **negative** : the system is bound !
  its value is equal to either

    minus the (+ve) Kinetic Energy, or
    half the (-ve) Potential Energy

- Here is a very useful little diagram to illustrate the situation : [image]

- Briefly reviewing the conditions necessary to use these simple equations :
    the system must be **self gravitating**
    the system must be in **steady state** (orbit timescale << evolution timescale)
    quantities must be **time averaged** (or many objects sampled with random orbital phase)
    the system must be **isolated** (or at least embedded in a slowly varying potential)
    Note that the system may be either collisionless (stellar) or collisional (gaseous)

(c) Mass Determination

- The most famous use of the virial theorem is to determine the masses of stellar systems.
  For a system of total mass M and mean squared velocity $<v^2>$, K is simply ½ M $<v^2>$
  The virial theorem then gives :
  $<v^2> = -W / M \equiv GM / R_g$
  which in practice defines the **gravitational radius**: R_g

  Knowing R_g and measuring $<v^2>$ allows us to determine M, the system mass.
  What to use for R_g isn't obvious for most stellar systems with no clear "edge" or "size"
  However, we can make use of the **median radius** : R_m which encloses half the mass
  For many stellar systems, it turns out that $R_g \simeq R_m / 0.4$ (note R_m is written r_h in B&T)
  We then have :

$$M_{tot} \simeq \frac{<v^2> R_m}{0.4 G} \tag{8.16}$$

  which resembles the circular orbit relation: $M = V^2 R / G$, but applies to a general self-gravitating system.

(d) Binding Energy : Energy Released During Collapse

- If the system **starts** very spread out and at rest : $E = K = W = 0$
  After settling down, we have once again : $E = K + W = -K$
  → energy must be **released** if the system collapses
  → this is termed the **binding energy**, and is the amount needed to unbind the system
  → the value of the binding energy is equal to the **remaining** KE
  → the total **gravitational** energy released is -W, of which
  half goes into KE, and half escapes the system
  Here is another little diagram to illustrate the situation : [image]

- Examples :
  - Collapsing protostars are luminous → they radiate half their gravitational potential energy
  - Kelvin considered a gravitational origin for the Sun's energy, via gradual contraction
  - For a galaxy, $K \sim ½ M_g V_c^2 \sim 10^{50}$ J $\equiv 10^{10} L_\odot \times 10^7$ years,
    this is 3×10^{-7} of the rest mass, ie $(V_c^2 / c^2) \times Mc^2$
    this is negligible in galaxy/starburst formation (nuclear burning is $\sim 7\times10^{-3}$ Mc^2)

(e) Stellar Systems Have Negative Specific Heat

Because gravitational energy is negative, bound systems have negative specific heat:
- Try to slow Earth's orbital motion by pulling back (i.e. **remove** orbital energy),
  it falls in to lower orbit and speeds up!
- Collapsing gas cloud radiates energy, collapses further, and **heats up**.
- Add energy to a star cluster (e.g. by accelerating the stars):
  the cluster expands and cools.

Here are diagrams to illustrate the situation : [image]

(f) Rotational Flattening

- Consider an axisymmetric system rotating about the z axis
  By symmetry :
  T, Π, and W are all diagonal
  x & y elements of these tensors are the same

- The tensor virial theorem gives :
  $2 T_{xx} + \Pi_{xx} + W_{xx} = 0$
  $2 T_{zz} + \Pi_{zz} + W_{zz} = 0$

- We also have :
  $T_{zz} = 0$ (rotation about z → no drift ∥ to z)

$2\,T_{xx} = ½ \int \rho <V_\phi>^2 d^3\mathbf{r} = ½\, M\, V_o^2$ (V_o is the mass weighted rotation speed)

$\Pi_{xx} = M\, \sigma_o^2$ (σ_o is the mass weighted dispersion)

$\Pi_{zz} \equiv (1 - \delta)\, \Pi_{xx} = (1 - \delta)\, M\, \sigma_o^2$ ($\delta < 1$, measures anisotropy)

$W_{xx} / W_{zz} \approx (A/B)^{0.9} = (1 - \epsilon)^{-0.9}$ (A/B is axis ratio of isodensity surfaces)

- Finally, substituting all these into the ratio of the two tensor relations above, we get :

$$\frac{V_o}{\sigma_o} = \sqrt{2\,[(1-\delta)(1-\epsilon)^{-0.9} - 1]} \qquad (8.17a)$$

B&T-1 fig 4.5 shows this relation for several δ, including projection corrections [images]
for isotropic velocities, $\delta = 0$, and we get, for small ϵ :

$$\frac{V_o}{\sigma_o} \simeq \sqrt{\frac{\epsilon}{(1-\epsilon)}} \qquad (8.17b)$$

- In this case, the inclination corrections to V_o / σ_o and ϵ are similar, so the prediction is robust

- Observationally, in Topic 7 we found (B&T-1 fig 4.6; [images]
  → Low luminosity Ellipticals and Bulges follow the isotropic relation
  → Luminous Ellipticals often fall in the anisotropic ($\delta > 0$) region

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(6) Describing Collisionless Systems

We first consider **collisionless** dynamics :

"Collision", here, means star-star **deflection**, not direct impact
For the collisionless case, stars are assumed to move in a **completely smooth** potential
For galaxies this is **almost always** a very good approximation

(in § 8.10 we consider when and how star-star encounters are relevant)

(a) The Distribution Function (DF) : f(r, v, t)

- A system is fully described by its **distribution function** (DF) or **phase space density** :
  $f(\mathbf{r}, \mathbf{v}, t)\,d^3\mathbf{r}\,d^3\mathbf{v}$ = number of stars at **r** with **v** at time t in range $d^3\mathbf{r}$ and $d^3\mathbf{v}$

- Knowledge of the DF is a holy grail, since it yields complete information about the system
  In practice, however, we only observe certain **projections** of the DF (eg $\Sigma(R)$, $V_p(R)$, $\sigma_p(R)$)

- Recovering the DF directly from observations is essentially impossible.
  To proceed, we need to introduce **further constraints** on the DF :
  an obvious example is $f(\mathbf{r}, \mathbf{v}, t) > 0$ everywhere and always, ie we cannot have -ve # stars !

  However, there are other constraints :

(b) Collisionless Boltzmann (Vlasov) Equation (CBE)

- Look for a **continuity equation**, since :
  no stars created/destroyed : flow conserves stars
  stars do not **jump** across the phase space (ie no **deflective** encounters)
  View the DF as a moving fluid of stars in 6-D space (**r, v**), ie x, y, z, v_x, v_y, v_z
  stars move/flow through the region as their positions and velocities change

- Consider a 1-D example using x and v_x, and recall f is a number **density**
  focus on a small element of phase space at x and v_x with size dx by dv_x

this [image] will help vizualise the situation

- In interval dt, net flow in x is :

$$v_x dt\, dv_x \left[f(x, v_x, t) - f(x + dx, v_x, t)\right] = -\, v_x\, dt\, dv_x \frac{\partial f}{\partial x}\, dx \qquad (8.18a)$$

the net flow due to the velocity gradient is

$$dx \frac{dv_x}{dt} dt \left[f(x, v_x, t) - f(x, v_x + dv_x, t)\right] = -\, dx\, dt \frac{dv_x}{dt} \frac{\partial f}{\partial v_x} dv_x \qquad (8.18b)$$

the sum of these equals the net change to f in the region, ie at x, v_x of size dx dv_x

$$dx\, dv_x \frac{\partial f}{\partial t} dt = -\, dt\, dx\, v_x \frac{\partial f}{\partial x} dv_x - dx\, dt \frac{dv_x}{dt} \frac{\partial f}{\partial v_x} dv_x \qquad (8.18c)$$

or, dividing by dx dv_x dt, we get

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + \frac{dv_x}{dt} \frac{\partial f}{\partial v_x} = 0 \qquad (8.19a)$$

but since

$$\frac{dv_x}{dt} = a_x = -\frac{\partial \Phi}{\partial x} \qquad (8.19b)$$

we have

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} - \frac{\partial \Phi}{\partial x} \frac{\partial f}{\partial v_x} = 0 \qquad (8.19c)$$

adding the y and z dimensions, which are independent, we finally have

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \nabla \phi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \qquad (8.19d)$$

This is the **collisionless Boltzmann equation** (CBE)

- The CBE describes how the DF changes in time
  It is a direct consequence of :

  1 conservation of stars
  2 stars follow smooth orbits
  3 flow of stars through **r** defines implicitly the location **v** (= d**r**/dt)
  4 flow of stars through **v** is given explicitly by $-\nabla\Phi$

- Since $\partial f/\partial t$ is a **Eulerian (partial)** differential, it describes the change in DF **at a point in phase space**

- However, consider the **Lagrangian (total, or convective)** derivative : $Df/Dt \equiv df/dt$.
  This describes the change in f as we follow **along the "orbit" through phase space**
  But, this Lagrangian derivative is nothing more than **the LHS of the CBE**

$$\frac{df}{dt} = \frac{\partial f}{\partial t}\frac{dt}{dt} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial v_x}\frac{dv_x}{dt} = 0 \qquad (8.20)$$

Clearly, the phase space density (f) along the star's orbit **is constant**

ie the flow is "incompressible" in phase-space
for example

if a region gets more dense, σ will increase
if a region expands, σ will decrease
example -- marathon race : start : n high, Δv high ; end : n low, Δv low

- The CBE applies to all **sub-populations** of stars (eg each spectral class)
  even though no single class determines the potential
  in § 8.7b we introduce a self-consistent f which **itself** generates Φ : [link]

(c) The Jeans Equation(s)

- As it stands, the CBE is of rather limited use :
  - the constraints it provides are still insufficient to find f(**r**,**v**,t)
  - the complexity of f(**r**,**v**,t) renders it observationally inaccessible.

- What we **observe** are :
  - **mean** velocities : < v >
  - velocity **dispersions** : σ (which is related to < v^2 >)
  - stellar **densities** : n (also ρ for mass density, or j for luminosity denisty)

  We need to recast the CBE in terms of these quantities.

- Clearly, these observable quantities are contained within the DF : f (**r**,**v**,t)
  they can be extracted by taking appropriate **averages** or **moments**
  for example :

  number density = n(**r**, t) = $\int$ f(**r**, **v**, t) d^3v = 0th moment in v
  mean velocity = <v_i(**r**, t)> = (1/n) $\int$ v_i f(**r**, **v**, t) d^3v = 1st moment in v

  If we take moments of the CBE, we transform it into equations in these new variables.
  Lets look in more detail at these first two moments in v (see B&T-2 §4.8) :

- Using the 1-D x axis as example, simply integrate the CBE (eq 8.19c) over all v_x
  We obtain (0th moment in v_x) :

$$\frac{\partial n}{\partial t} + \frac{\partial\left(n\langle v_x\rangle\right)}{\partial x} = 0 \qquad (8.21)$$

  where n $\equiv$ n(x,t) is the space density and <v_x> is the mean drift velocity along x
  This is a simple continuity equation for the number of stars along the x axis.

- Now multiply the CBE (eq 8.19c again) through by v_x and again integrate over all v_x
  on rearranging and using eq 8.21 above, we obtain (1st moment in v_x) :

$$\frac{\partial\langle v_x\rangle}{\partial t} + \langle v_x\rangle\frac{\partial\langle v_x\rangle}{\partial x} = -\frac{\partial\Phi}{\partial x} - \frac{1}{n}\frac{\partial(n\sigma_x^2)}{\partial x} \qquad (8.22a)$$

  where σ_x^2 is the velocity dispersion about the mean velocity,
  it arises from <v_x^2> = <v_x>^{2} + σ_x^2

- repeating this in 3-D requires a little care (B&T-2 § 4.8) :
  we obtain the **Jeans Equation** (for coordinate j) :

$$\frac{\partial\langle v_j\rangle}{\partial t} + \langle v_i\rangle\frac{\partial\langle v_j\rangle}{\partial x_i} = -\frac{\partial\Phi}{\partial x_j} - \frac{1}{n}\frac{\partial(n\sigma_{i,j}^2)}{\partial x_i} \qquad (8.22b)$$

  where the summation convention applies (sum over repeated indices)
  here, i=1,2,3 and j=1,2,3 refer to x,y,z, eg $x_2 \equiv$ y and $v_2 \equiv v_y$

- This Jeans equation is akin to Newtons's second law : dv/dt = F/m with :
  LHS is the derivative of <v>
  RHS are force terms

- It is instructive to compare this to **Euler's Equation for fluid flow** :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\nabla\Phi - \frac{1}{\rho}\nabla p \qquad (8.23)$$

  which is clearly analogous.

- In 8.22b n $\sigma_{i,j}^2$ is a **stress tensor** which takes the role of an **anisotropic pressure**
  (hence the phrase "pressure supported")
  in a fluid, pressure is a **scalar** and is therefore always **isotropic**
  for stellar systems, $\sigma_{i,j}$ is a **tensor** which can be **anisotropic**

- $\sigma_{i,j}$ is **symmetric,** : i.e. axes exist where $\sigma_{1,1}, \sigma_{2,2}, \sigma_{3,3}$ are semi-axes of a velocity ellipsoid
  if $\sigma_{1,1} = \sigma_{2,2} = \sigma_{3,3}$ we have **isotropic** dispersion → Jeans and Euler equations are identical

- For collisionless systems there is no **equation of state** linking pressure ($\sigma_{i,j}^2$) to density
  Usually, therefore, we are forced to **assume** $\sigma_{i,j}$ (or, equivalently, the anisotropy parameter β)
  Recently, however, the LOSVD has been used to constrain β (see T 5.7a : [link]).

(d) Applications of the Jeans Equation

The Jeans equation, when combined with observations, has a number of applications :

-- deriving M/L profiles in spherical galaxies (B&T-1 4.2.1d)
-- deriving the flattening of a rotating spheroid with isotropic velocity dispersion (B&T-1 4.2.1e)
-- analysis of asymmetric drift (B&T-1 4.2.1a)
-- surface density (and volume density) in the galactic disk (B&T-1 4.2.1b)
-- analysis of the local velocity ellipsoid in terms of Oort's constants (B&T-1 4.2.1c)

Here we look briefly at the first and second :

(i) Spherically Symmetric Steady State Systems

- This is, of course, an important special case to consider :
  For steady state, the first term in Eq 8.22b is zero
  For spherical symmetry : $\langle v_r \rangle = \langle v_\theta \rangle = 0$, giving $\langle v_r^2 \rangle = \sigma_r^2$ and $\langle v_\theta^2 \rangle = \sigma_\theta^2$.

  After transforming to spherical polar coordinates, the Jeans Equation reads :

$$\frac{1}{n}\frac{d(n\sigma_r^2)}{dr} + \frac{1}{r}\left[2\sigma_r^2 - (\sigma_\theta^2 + \sigma_\phi^2)\right] - \frac{\langle v_\phi \rangle^2}{r} = -\frac{d\Phi}{dr} \qquad (8.24a)$$

  Introducing **anisotropy parameters** : $\beta_\theta = 1 - \sigma_\theta^{\ 2} / \sigma_r^2$ and $\beta_\phi = 1 - \sigma_\phi^{\ 2} / \sigma_r^2$
  and writing 2β for $\beta_\theta + \beta_\phi$ and V_{rot} for $\langle v_\phi \rangle$ this becomes

$$\frac{1}{n}\frac{d(n\sigma_r^2)}{dr} + 2\beta\frac{\sigma_r^2}{r} - \frac{V_{\rm rot}^2}{r} = -\frac{d\Phi}{dr} \qquad (8.24b)$$

  which is equivalent to the equation of hydrostatic support :
  dp /dr + anisotropic correction + centrifugal correction = F_{grav}

- Going a little further, recasting $d\phi / dr$ as $GM(<r) / r^2 = V_c^2 / r$ (V_c = circular velocity)
  and rewriting the first term in eq 8.24b in logarithmic gradients, we have :

$$V_{\rm rot}^2 - \sigma_r^2\left(\frac{d\ln n}{d\ln r} + \frac{d\ln(\sigma_r^2)}{d\ln r} + 2\beta\right) = \frac{GM(<r)}{r} = V_c^{\,2} \qquad (8.24c)$$

This parallels the equation for hydrostatic support of an ideal gas, where p = nkT
the equivalences are :

$\sigma_r^2 \equiv T$

$d(\ln n) / d(\ln r) + d(\ln T) / d(\ln r) \equiv (n/p)\, dp / dr$

2β and V_{rot}^2 are anisotropy and rotation correction terms

- By measuring brightness profiles and velocity dispersion & rotation profiles,
we can derive (**assuming** β) : M(r) and hence M/L (r)
This is very important, eg, in the search for nuclear black holes (see Topic 14.2 : link)

(ii) Rotational Flattening Revisited.

TBD

(iii) Vertical Disk Structure.

TBD

(7) Steady State : The DF as f(E, |L|, L_z)

Taking moments of the CBE lost almost all detailed information from the DF
Rather than working with the full DF, the Jeans equation works with just n, <v> and <v^2>
Can we reintroduce the full DF and regain a more complete description of a system ?

The answer is **yes**, by introducing two new powerful constraints :
→ demand that the system is in **steady state** (≡ in equilibrium)
→ demand that the DF **generate the full potential** (not just act as a tracer population)

We consider these in turn

(a) Integrals of Motion and the Jeans Theorem

- When a system is in steady state, Φ and f are not **explicit** functions of time
In this case, we may introduce a powerful new entity : **Integrals of motion**
An "integral of motion" is a function I (**r**, **v**) which is **constant** along a star's orbit (B&T-1 § 3.1.1)
Obvious examples of possible integrals of motion are :

E (**r**, **v**)	=	$\frac{1}{2}\mathbf{v}^2 + \Phi(\mathbf{r})$	=	energy per unit mass	in a **static potential**
L (**r**, **v**)	=	$\mathbf{r} \times \mathbf{v}$	=	total AM	in a **spherical static potential**
L_z (**r**, **v**)	=	$(x^2 + y^2)^{1/2}\, v_\phi$	=	z component of AM	in an **axisymmetric static potential**

- Since I (**r**, **v**) is constant along an orbit, it is also a solution to the steady state CBE
specifically :

$$\begin{aligned} \frac{dI}{dt} &= \sum_{i=1}^{3} \frac{\partial I}{\partial x_i}\frac{dx_i}{dt} + \sum_{i=1}^{3} \frac{\partial I}{\partial v_i}\frac{dv_i}{dt} = 0 \\ &= \nabla I \cdot \frac{d\mathbf{x}}{dt} + \frac{\partial I}{\partial \mathbf{v}} \cdot \frac{d\mathbf{v}}{dt} = 0 \\ &= \mathbf{v} \cdot \nabla I - \nabla \Phi \cdot \frac{\partial I}{\partial \mathbf{v}} = 0 \end{aligned} \quad (8.25)$$

- Since the CBE is a linear equation, then functions of solutions are themselves solutions
This yields the **Jeans Theorem** :

Any function of integrals of motion f (I_1, I_2, I_3, ...) **is also a solution of the steady state CBE**

- This is **extremely useful** since it allows us to construct legitimate DFs using integrals of motion : eg,: the DF : $f(E, L_z) = N_o (E^2 + 3L_z^{5/2})$ is a solution to the CBE for an axisymmetric potential

- In the special case of steady state **spherical systems**, it is easy to show (B&T-1 § 4.4.2) that :
  - DFs must have the form f(E, |L|)
  - DFs of the form f(E) must have an **isotropic** velocity dispersion $\sigma_r = \sigma_\theta = \sigma_\phi$
  - DFs of the form f(E, |L|) must have an **anisotropic** velocity dispersion $\sigma_r \neq \sigma_\theta = \sigma_\phi$

- Summarizing: these theorems provide a very useful way to begin constructing working models :
  For each **r** and **v** location in phase space calculate, for example, E, |L|, L_z
  Now assign the number of stars at that location in phase space, f(**r**,**v**), by some function of E, |L|, L_z.
  These DFs now **automatically** satisfy the continuity condition expressed by the steady state CBE.

(b) Self-Consistency

- Both the CBE and the Jeans Equation include a potential gradient, $\nabla\Phi$
  In neither equation, however, are these potentials linked explicitly to the DF
  (recall $\int f(\mathbf{r}, \mathbf{v})\, d^3v = n(\mathbf{r}) \equiv \rho(\mathbf{r})$ which could, in principle, define Φ)
  As it stands, the DFs only describe **tracer** populations.

- Clearly, an important step is to **require** that the DF **also** yields the potential $\Phi(\mathbf{r})$
  ie :

$$4\pi G \int f(\mathbf{r}, \mathbf{v})\, d^3\mathbf{v} = 4\pi G \rho(\mathbf{r}) \qquad (8.26a)$$

$$= \nabla^2 \Phi(\mathbf{r}) \qquad (8.26b)$$

  where f here is the mass DF (ie we've multiplied f by the mean stellar mass)

- Taking the spherical form for ∇^2, this reads (eg for a DF of the form f (E, |L|) :

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi}{dr}\right) = 4\pi G \int f(\tfrac{1}{2}v^2 + \Phi\, ;\, |\mathbf{r}\times\mathbf{v}|)\, d^3\mathbf{v} \qquad (8.27)$$

  This is now a fundamental equation describing spherical equilibrium systems.
  Solutions not only have self consistent Φ and f, but f also satisfies the steady state CBE.
  Such a solution now describes a self-consistent, physically plausible stellar dynamical system.

- When using this equation to solve the structures of many systems, we introduce (B&T-1 § 4.4) :
  - **relative potential** : $\Psi = \Phi_o - \Phi$
  - **relative energy** : $E_r = -E + \Phi_o = \Psi - \frac{1}{2} v^2$
  - note : both Ψ and E_r are more +ve for more bound stars deeper in the system
  - choose Φ_o so that f > 0 for $E_r > 0$ (bound)
  - at given Ψ : E_r spans range 0 to Ψ, as v spans the range from $\sqrt{2\Psi}$ (= V_{esc}) to 0

(c) Spherical Isotropic Systems : DF = f(E_r)

- If we take f = f(E_r) and adopt the variables above, eq 8.27 takes the form [recall $d^3v = 4\pi v^2\, dv$] :

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Psi}{dr}\right) = -16\pi^2 G \int_0^{\sqrt{2\Psi}} f(\Psi - \tfrac{1}{2}v^2)\, v^2\, dv \qquad (8.28a)$$

$$= -16\pi^2 G \int_0^{\Psi} f(E_r)\, \sqrt{2(\Psi - E_r)}\, dE_r \qquad (8.28b)$$

  These now describe a spherical, non-rotating, isotropic velocity dispersion system.
  They will be our starting point in constructing specific spherical models in § 8.8

(d) Deriving f(E_r) from ρ(r) for Non-Rotating Spherical Systems

- The above method starts by choosing a DF, then uses eq 8.28 to calculate $\rho(r)$
  In practice, however, we can often **measure** $\rho(r)$ from (deprojected) surface photometry.
  Is it possible to reverse the method and derive $f(E_r)$ from a known density profile ?
  The answer is **yes**

- First evaluate $\Psi(r) = -\Phi(r) = GM(<r) / r$ from $\rho(r)$ and eliminate r to find $\rho(\Psi)$
  we then find $f(E_r)$ from the **Eddington (1916) Formula** (B&T-1 4.4.3d) :

$$f(E_r) = \frac{1}{\pi^2\sqrt{8}}\left[\int_0^{E_r} \frac{d^2\rho}{d\Psi^2}\frac{d\Psi}{\sqrt{E_r-\Psi}} + \frac{1}{\sqrt{E_r}}\left(\frac{d\rho}{d\Psi}\right)_{\Psi=0}\right] \quad (8.29)$$

- This can be done for any $\rho(r)$ though one must be careful that $f(E_r) > 0$ at all E_r
  examples are : deVaucouleurs $R^{1/4}$ law & Jaffe law [image] (B&T-1 fig 4.12)

- This method can be extended to rotating spherical systems with f(E, |L|) : B&T-1 Eq. 4-149
  as well as axisymmetric systems with $f(E, L_z)$ and $f(E, L_z, I_3)$: B&T-1 §4.5.2a and §4.5.3.

(e) From $f(E_r)d^3r\, d^3v$ to $N(E_r)dE$

- For N-body simulations, it is often useful to evaluate $N(E_r)\, dE_r$:
  i.e. the total number of stars as a function of energy, E_r.
  Note that $N(E_r)$ is **not** simply the DF $f(E_r)$ since this describes the number of stars
  of energy E_r **at each point in phase space r, v** in the range $d^3r\, d^3v$
  while $N(E_r)$ is the total number within the system of energy E_r in the range dE_r.

- Integration of $f(E_r)$ gives (B&T-1 4.4.5) :

$$N(E_r)\,dE_r = 16\pi^2 f(E_r) \int_0^{r_m(E_r)} r^2 \sqrt{2(\Psi(r)-E_r)}\,dr \quad (8.30)$$

  where r_m = largest radius out to which a star with E_r can be found i.e. v=0 at $\Psi(r_m) = E_r$

- While $f(E_r)$ typically **increases** exponentially with E_r
  $N(E_r)$ typically **decreases** with E_r, with a maximum near $E_r \sim 0$ (where $f(E_r)$ is usually small)
  → Most stars are nearly unbound ($E_r \sim 0$)
  → Few stars are deeply bound ($E_r \sim \Psi(r=0)$)

- Examples of N(E) dE (note, not E_r) for the deVaucouleurs, King, and two Jaffe models :
  (B&T fig 4.15) [image]

Next Prev Top

(8) Model Building Using DFs

We begin with the simplest cases : equilibrium, non-rotating, spherical systems, ie DF $\equiv f(E_r)$
With equations 8.28a,b now in hand, we are ready to construct specific models
The process goes as follows :

(1) Choose a DF which is a function of energy : $f(E_r) \equiv f(\Psi - \frac{1}{2}v^2)$
from Jeans Theorem, $f(E_r)$ is already a solution to the steady state CBE,
so our solutions will naturally satisfy the basic phase space continuity condition
(2) Integrate the DF over v to find $\rho(\Psi)$ (ie evaluate 8.26a)
(3) Solve Poisson's equation (8.28a) to find $\Psi(r)$
(4) Combine $\rho(\Psi)$ and $\Psi(r)$ to give the mass distribution : $\rho(r)$

Here are some examples

(a) Polytropic Sphere: Power Law $f(E_r)$

- Consider a **power law** DF : $f(E_r) = F\, E_r^{n-(3/2)}$ for $E_r > 0$ (otherwise $f(E_r) = 0$)

  Integrate $f(E_r)$ over velocity to find the density in terms of Ψ (eq 8.26a) :

$$\rho = 4\pi \int_0^\infty f(E_r)\, v^2\, dv = 4\pi F \int_0^{\sqrt{2\Psi}} (\Psi - \tfrac{1}{2}v^2)^{n-\frac{3}{2}} v^2\, dv \qquad (8.31)$$

  after substituting $v = (2\Psi)^{½}\cos\theta$, we find $\rho(\Psi) = c_n\, \Psi^n$ ($\Psi > 0$)
  where c_n is a constant depending on n and F.

- Substitute this into the spherical version of Poisson's equation (eqn 8.28a) :

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Psi}{dr}\right) + 4\pi G c_n \Psi^n = 0 \qquad (8.32)$$

- This is the Lane-Emden equation, first studied as the equation describing hydrostatic equilibrium of a self-gravitating sphere of polytropic gas (ie equation of state : $p \propto \rho^\gamma$)
  Thus, we find that for a self-gravitating sphere, the density profile $\rho(r)$ is the **same** for
  stars with DF $\propto E_r^{n-(3/2)}$, and
  gas with polytropic equation of state and $\gamma = 1 + (1/n)$

- Simple solutions only exist for n = 5 ($\gamma = 6/5$)
  this is the **Plummer Sphere** with $\rho(r) \propto (1 + (r/b)^2)^{-5/2}$
  it has finite mass and is well behaved at r = 0
  it is a good match to Globular Clusters but is too steep at large r for Ellipticals

- n > 5 systems are more extended and have infinite mass
  density profiles for n=0,1,2,3,4,5 are shown here : [image]
  density; potential; rotation & image for a Plummer sphere are shown here : [image]

- $n = \infty$ so $\gamma = 1$ and $p \propto \rho$ which is the **isothermal** equation of state (recall P = n k T)
  for $n = \infty$ the above analysis breaks down, but we have an alternative approach :

(b) Isothermal Sphere: Exponential $f(E_r)$

- Consider an **exponential** (Boltzmann) DF

$$f(E_r) = \frac{\rho_1}{(2\pi\sigma^2)^{\frac{3}{2}}}\, e^{E_r/\sigma^2} \qquad (8.33)$$

  Recall, more +ve Ψ & E_r means more bound.
  Also, note $f(E_r) > 0$ for $E_r < 0$: there are unbound stars! .... we anticipate problems at large radii.

  OK, substituting Ψ - ½v^2 for E_r and integrating $f(E_r)$ over v gives $\rho = \rho_1 \exp(\Psi / \sigma^2)$

- Plugging this into Poisson's equation gives :

$$\frac{d}{dr}\left(r^2 \frac{d\ln\rho}{dr}\right) = -\frac{4\pi G}{\sigma^2} r^2 \rho \qquad (8.34)$$

  This is, in fact, the equation for a hydrostatic sphere of isothermal gas, with $\sigma^2 = kT/m$
  Why is this ?
  At every point, $N(v) \propto \exp(-½v^2/\sigma^2)$, for both the stellar system and a gas of atoms
  it is irrelevant, therefore, whether the stars are collisionless or not, they mimic a gas of atoms.

- Traditionally, we consider the solutions to 8.34 as (i) "a special case" and (ii) "the rest" :

(i) Singular Isothermal Sphere (SIS)

- For the central boundary condition $\rho(0) = \infty$ we have $\rho(r) = \sigma^2 / (2\pi G r^2)$
  this is the **singular isothermal sphere**: $\rho \propto r^{-2}$
- Circular velocity : $V_c = \text{const} = \sigma\sqrt{2}$
- Dispersion velocity : $\langle v^2 \rangle = 3\sigma^2$ everywhere (isothermal !); 1-D : $\langle v_r^2 \rangle = \sigma^2$
- But the model has infinite density at r = 0, and has infinite mass as $r \to \infty$!
- Density; potential; rotation & image for SIS are shown here : [image]

(ii) General Isothermal Sphere

- Choose as central boundary conditions at r = 0 :
  $\rho(0) = \rho_o$ finite central density
  $(d\rho/dr)_{r=0} = 0$ flat central density profile
  Integration of 8.34 with these boundary conditions yields $\rho(r)$ [image]: B&T-1 figs 4.7, 4.8

- We find a **constant** near-nuclear density : $\rho(r) \sim \rho_o$ within a radius $r_o = 3\,\sigma / (4\pi G \rho_o)^{½}$
  This is a **core** and r_o is called the King (or core) radius
  $I(r_o) = 0.5013\, I(0)$, so r_o is appropriately defined
  r_o is also the scale length of the r^{-2} envelope (see below): big cores are in big galaxies
  Circular velocity : $V_c = -\sigma\,(d \ln \rho / d \ln r)^{½}$

- When plotted as $\log(\rho / \rho_o)$ vs $\log(r / r_o)$, there is only **one** isothermal profile

  - At **small** radii (e.g. r < few r_o)
    the **density law** resembles the Hubble density law : $\rho(r) \approx (1 + (r / r_o)^2)^{-3/2} = \rho_H(r)$
    → I(R) fits ~OK to the centers of many Elliptical galaxies

  - At **large** radii (e.g. $r \gtrsim 15\, r_o$)
    the system resembles the SIS : $\rho(r) \propto (r / r_o)^{-2}$ and $V_c = \sigma\sqrt{2}$
    this is **different** from the Hubble density Law:
    → projected light profile **does not** fit Ellipticals well in the outer parts (too flat)

- The scale length and central density together define the dispersion : $\sigma^2 \propto \rho_o r_o^2$
  → for a given central density, hotter galaxies are larger
  → for a given core radius, hotter galaxies are denser

- Quantitatively : $\sigma^2 = (4 / 9)\, \pi G \rho_o r_o^2$
  To simplify calculations, use $G = 4.5 \times 10^{-3}$ in units of pc, km/s, and $M_\odot$
  Eg for σ = 100 km/s, r_o = 100 pc we have $\rho_o = 159\ M_\odot\ pc^{-3}$

- A good isothermal core match to the centers of Ellipticals can be used to estimate central M/L
  → obtain r_o and I(0) from isothermal fits to I(R), and measure σ
  (express I(0) in units of $L_\odot\ pc^{-2}$ to allow simplified calculations with $G = 4.5 \times 10^{-3}$)

  $j(0) = 0.5\, I(0) / r_o$
  $\rho(0) = 9\,\sigma^2 / (4\pi G r_o^2)$
  $M/L = \rho(0) / j(0)$

  This method is called "core fitting" or "King's method"
  Typical values for ellipticals cores are $\simeq$ 10-20 h $M_\odot / L_\odot$ suggesting minimal/no dark matter

- There is a problem with all isothermal models: **they have infinite total mass**
  It is easy to see why the system is at least infinite in extent :
  at any given radius, stars have isotropic dispersion σ
  at this radius at least some stars are therefore moving **outward**
  but further out the dispersion is **still** σ, and stars are moving outward
  → the system must have infinite extent
  Ultimately, this arises because $f(E_r) > 0$ for negative E_r, i.e. the model includes **unbound stars**.

To rectify this problem, we attempt to modify things slightly by removing the unbound stars: →

(c) Lowered Isothermal (King): Truncated Exponential $f(E_r)$

- Suppress stars at large radius (ie as $E_r \to 0$, we want $f(E_r) \to 0$)
  modify the exponential DF :

$$f(E_r) = \frac{\rho_1}{(2\pi\sigma_0^2)^{\frac{3}{2}}} \left(e^{E_r/\sigma_0^2} - 1\right) \quad (8.35)$$

  where σ_o is a (dispersion like) parameter.

- Repeating the same analysis as before, we get for Poisson's eqn :

$$\frac{d}{dr}\left(r^2 \frac{d\Psi}{dr}\right) = -4\pi G \rho_1 r^2 \left[e^{\Psi/\sigma_0^2} \mathrm{erf}\left(\frac{\sqrt{\Psi}}{\sigma_0}\right) - \sqrt{\frac{4\Psi}{\pi\sigma_0^2}}\left(1 + \frac{2\Psi}{3\sigma_0^2}\right)\right] \quad (8.36)$$

  Solve this by integration, choosing boundary conditions at r = 0 :

  $\Psi(0) = q\,\sigma_o^2$ (q > 0, large q = deep central potential)
  $d\Psi / dr = 0$ (as before)

- Inner regions : like isothermal, with core (King) radius ~ r_o (defined as before)
  Outer regions : $\Psi(r)$ decreases & approaches 0 at r_t
  recall : velocity **range** at Ψ is 0 → $\sqrt{(2\Psi)}$
  so density = $\int f\, d^3v = 0$ at r_t = **tidal** or **truncation radius** = **edge** of sphere
  larger $\Psi(0)$ (larger q) → larger r_t & $M_{tot} \equiv M(r_t)$
  Alternative parameter to $\Psi(0)$ or q is **concentration** $c = \log_{10}(r_t / r_o)$
  ([images] : B&T fig 4.9, 4.10, 4.11)

- Single sequence of King models by varying (equivalently) : $\Psi(0)$; q; c
  ([images] : B&T fig 4.9)
  Empirically, we find :
  c = 0.75 - 1.75 (≡ q = 3 - 7) fit GCs very well
  c > 2.2 (≡ q > 10) fit some Ellipticals quite well
  c = 1.7 (≡ q = 8) fits Hubble law well
  c = ∞ (≡ q = ∞) is the isothermal sphere

- King models are **not** isothermal : $\sigma^2 \equiv \langle v^2 \rangle \simeq \sigma_o^2$ within r_o but drops at larger radii
  ([images] : B&T fig 4.11)
  However, as with isothermal models, for each c (or q) we have a range of King models
  each of different σ_o, subject to $\sigma_o^2 \propto \rho_o\, r_o^2$
  eg for given r_o, high ρ_o has high σ_o

(d) Other Models

The methods illustrated here can be applied to more complex systems:
Spherical systems with velocity anisotropy (B&T-2 4.3.2)
Axisymmetric systems (B&T-2 4.4)
Thin disks (B&T-2 4.5)

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(9) Violent Relaxation

- The previous discussion focussed on **static systems**, since they are relatively tractable.

Varying potentials are usually intractable and require a numerical approach.
There are, however, a few situations which can be treated analytically
Paradoxically, one of these is when the potential is maximally fluctuating
This is the case of **violent relaxation**, which we now describe briefly

- For galaxies, 2-body encounters are negligible and evolution is determined by the CBE
For a static potential, energy (E) of a star is conserved and the DF doesn't change
Isolated galaxies in steady state do **not**, therefore, evolve dynamically
(we're ignoring gas & 2-body processes here)

- For a galaxy to change, there needs to be a **changing potential**
For each star, $dE / dt = \partial\Phi / \partial t$ at the star
The DF evolves and the structure of the galaxy changes
This occurs during (i) inhomogeneous collapse, and (ii) encounters (Topic 12)
These are brief traumatic times :
→ "galaxy changes are by **revolution** rather than by **evolution**"
(nice quote from Binney's EAA article)

- In collapse of large cold system, Φ changes rapidly
stars gain and lose energy, which broadens f(E)
energy is **redistributed** via **collective interactions**
this acts like a **relaxation process** (example movie).

- Note : the **total** energy remains constant : this is a **non-dissipational** process
energy is **not** radiated away, as with dissipational (gaseous) collapse
If the total energy is initially zero (eg diffuse system at rest), then following collapse :
some stars will be strongly bound, but some must also have been ejected.

- Note : scattering is independent of the star's mass
fundamentally **different** from 2-body relaxation
no segregation by mass (eg heavy stars **don't** sink to center)

- **Phase mixing** helps smooth out lumps fairly quickly
distribution is ~smooth after ~few collapse times
→ violent relaxation timescale is ~few × dynamical (collapse) timescale

- If relaxation is **complete**, then fully random scattering occurs
→ results in isotropic velocity field and Boltzmann-like f(E)
Usually, however, the density distribution becomes smooth **before** scattering is complete
relaxation ceases and is **incomplete** → residual anisotropies & phase-space substructures
(< viewgraph >)

- N-Body example : van Albada 1982 (B&T 4.7.3) ([images] : B&T figs 4.19-23)
start with ~ homogeneous sphere with low σ
1st infall → dense center
settles into ~ $R^{1/4}$ law
σ drops with radius
β (anisotropy) is 0 at nucleus, → 1 at edge
(most scattering occurred at small r on 1st infall → most stars have low AM)
N(E) dE spreads out, most stars have E ~ 0, few are deeply bound

  If the initial distribution is hotter → less concentrated
  If the initial distribution is rotating slowly → less concentrated & rotating oblate figure
  If the initial distribution is rotating faster → even less concentrated & prolate/bar figure
  If the initial distribution is ellipsoidal → rotating ellipsoid, anisotropic everywhere

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(10) Introducing Star-Star Encounters

So far, we have considered star motion in a **perfectly smooth** potential
However, in reality, individual stars render this potential bumpy on fine scales

How does this affect the motion of stars --- ie is the "collisionless" assumption valid ?

(a) Estimating Encounter and Relaxation Timescales

- As usual : mean free path = 1 / nA and time between encounters = 1 / nAV
for encounter crossection A ~ b^2 (b = impact parameter); star density n; and mean velocity V
We consider three regimes :

(i) Direct collision (or tidal capture)

- For b ~ few × R_{star} strong tides dissipate orbital energy, leading to tidal capture
depending on circumstances, the stars may ultimately coalesce
→ this is **exceedingly rare** in present day galaxies

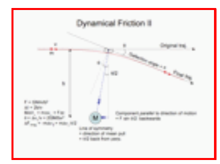
(ii) Strong Deflection

- Defined as $\Delta V \simeq V$ occurring when $b \equiv r_s$ (s for strong) is sufficiently small
from virial theorem : $G m^2 / r_s \sim m V^2$ so $r_s \sim G m / V^2$ (here m is star mass)
→ this is ~ 1 AU for the sun, using V ~ 20 - 30 km/s
(cf $V_{\oplus}$ for earth's orbit $\approx \sigma_*$ in solar neighborhood)

- The interval, t_s, between collisions = $1 / n r_s^2 V = V^3 / (G^2 m^2 n) \sim 10^{15}$ years for sun
→ this is **very rare** for most stellar systems
may be relevant in dense GC nuclei, galactic nuclei and galaxy clusters

(iii) Weak Deflection

- Defined as $\Delta V << V$ so $b >> r_s$
estimate deflection velocity towards target star : [image]
time in vicinity of target star : $\Delta t \approx 2b / V$
$\perp$ acceleration $\approx Gm / b^2$ so $\Delta V_{\perp} \approx 2Gm / bV$
Angle of deflection $\Delta\theta \approx \Delta V_{\perp} / V \approx 2Gm / bV^2 \approx 2$ arcsec in solar neighborhood



- After many encounters : $\mathbf{\Delta V_{tot}} = \sum \mathbf{\Delta V}$
since the **direction** of pull is **random**, $\mathbf{\Delta V_{tot}}$ executes a random walk
the amplitude (squared) of this resultant velocity after time t is given by :

$$|\Delta V_{\rm tot}|^2 = \sum |\Delta V|^2 = \int_{b_{min}}^{b_{max}} \left(\frac{2Gm}{bV}\right)^2 t\, nV\, 2\pi b\, db \quad (8.37a)$$

$$= \frac{8\pi G^2 m^2 n t}{V} \ln\Lambda \quad (8.37b)$$

where $\Lambda = b_{max} / b_{min}$

- The system has relaxed when the velocity changes by ~100%, ie when $\Delta V_{tot} \simeq V$
Integrating over a Maxwellian changes the numerical constant slightly (B&T §8.4)
Changing V to σ, and writing m n as ρ, the stellar density, we get (B&T eq 8-71)

$$t_{\rm relax} \simeq 0.34 \frac{\sigma^3}{G^2 m \rho \ln\Lambda} \quad (8.38a)$$

$$\simeq \frac{1.8 \times 10^{10}\ {\rm yr}}{\ln\Lambda} \sigma_{10}^3\, m_{\odot}^{-1}\, \rho_3^{-1} \quad (8.38b)$$

where σ_{10} has units 10 km/s, m has units of $M_{\odot}$, and ρ_3 has units $10^3\ M_{\odot} / pc^3$

- There is a surprisingly simple alternative expression for the relaxation time
It is less precise but is adequate in many circumstances

We start, as before, with equation 8.37b :

for a system of size R containing N stars : $n = 3N / (4\pi R^3)$

from the virial theorem : $V^2 = GM / R = GNm / R$

the system **relaxes** when $\Delta V_{tot} \simeq V$

take $b_{max} \simeq R$; $b_{min} \simeq r_s = Gm / V^2$, so $b_{max} / b_{min} = \Lambda = N$

choose units of time : $t = t_{cross} \approx R / V$

Substituting, we get

$$t_{relax} \simeq t_{cross} \frac{N}{6 \ln N} \quad (8.38c)$$

- Surprisingly, this only depends on N, the total number of stars in the system
- Notice that $t_{relax} > t_{cross}$ for $N \gtrsim 30$
  → to good approximation, stars usually orbit in the overall potential
- Equal logarithmic intervals in b contribute equally to long term deflection.
  eg the ranges : R to ½R ; ½R to ¼R ; .... $2b_{min}$ to b_{min} all contribute **equally**
- However, since the deflection drops rapidly with b as $\Delta V / V \propto 1 / b$
  → for systems with $R >> b_{min}$ most scattering is due to **weak encounters** ($\Delta V << V$)
  Example :
  galaxy with R ~ 10 kpc ; b_{min} ~ 1 AU (1 $M_\odot$ stars); so $\ln \Lambda = 20$
  half deflection from encounters outside b_1 where $\ln R/b_1 = 10$
  3/4 deflection from encounters outside b_2 where $\ln R/b_2 = 15$
  $b_1 = 0.5$ pc for which $\Delta V / V = b_{min} / b_1 \approx 10^{-5}$
  $b_2 = 0.003$ pc for which $\Delta V / V = b_{min} / b_2 \approx 0.15\%$
- Need care with N-Body simulations when $N << N_{stars}$
  Φ is more grainy than reality, and t_{relax}(simulation) << t_{relax}(reality)
  avoid by **softening** star potentials to increase b_{min}
  care : lose structure on scales $R < b_{min}$

(b) Timescales for Real Stellar Systems

- Here are rough timescales (in years) for a number of stellar systems :

System	**N**	**R** (pc)	**V** (km/s)	t_{cross}	t_{relax}	t_{age}	**age/relax**
Open Cluster	10^2	2	0.5	10^6	10^7	10^8	10
Globular Cluster	10^5	4	10	5×10^5	4×10^8	10^{10}	20
Dwarf Galaxy	10^9	10^3	50	2×10^7	10^{14}	10^{10}	10^{-4}
Elliptical	10^{11}	$10^{4.5}$	250	10^8	4×10^{16}	10^{10}	10^{-7}
Spiral Disk	10^{11}	$10^{4.5}$	20	1.5×10^9	6×10^{17}	10^{10}	10^{-8}
MW Nucleus	10^6	1	150	10^4	10^8	10^{10}	100
Luminous Nucleus	10^8	10	500	2×10^4	10^{10}	10^{10}	1
(Galaxy Cluster)	10^2	5×10^5	500	10^9	(3×10^9)	10^{10}	(3)

- The presence of dark matter complicates the situation in clusters (see [Topic 13.4c])
  In practice, 2-body relaxation is **not** as fast as our simple analysis suggests.
- 2-Body relaxation may be relevant for star clusters and galaxy nuclei

- For most galaxies, 2-body relaxation is utterly negligible.
  Because this course deals specifically with galaxies (and not star clusters)
  we will only briefly consider the ramifications of relaxation.

- Don't forget, relaxation times can vary greatly **within a given system**
  for example, a GC core can be relaxing while the halo is not

(c) Analytic Treatment : The Fokker-Planck Equation

- You may be wondering when Max Planck (& Adriaan Fokker) worked on stellar dynamics....
  They **didn't** : much of this work has its roots in **plasma physics**
  Unlike neutral gases, charges in plasmas have long range Coulomb interactions
  The early work on plasmas has been appropriated and applied to stellar systems

- For a smooth potential, the DF obeys the CBE : df / dt = 0
  with encounters, stars scatter into and out of the phase space trajectory : df / dt = Γ(f)
  Γ(f) is a **collision term** and itself depends on f

- If the full collision term is included we have the **master equation.**
  If most scatterings are distant, an approximation for the collision term yields the **Fokker-Planck equation.**
  This is a PDE, for which several approaches to solutions have been made (see B&T-2 7.4).

(d) Results : The Effects of Encounters

There are a number of distinct phenomena which result from 2-body encounters :

(i) Relaxation

- 2-body relaxation introduces the equivalent of thermal conduction in a gas
  For self-gravitating systems, this can be a rather interesting process
  recall from [§8.5e] that such systems have **negative specific heat**
  → if you **remove** energy (heat), stars fall deeper in the gravitational well
  → they therefore speed up, and that part of the system gets **hotter**

- In its simplest form, this relaxation renders clusters more centrally concentrated
  → stellar encounters in the core pass energy to envelope stars
  → the core contracts and heats, the envelope expands and cools
  after some time the envelope develops a density profile $\rho(r) \propto r^{-3.5}$
  radial anisotropy increases with time and radius
  (stars have been kicked out from encounters in the core and carry little AM)
  a successful DF is due to Michie, and is $f(E,L) \propto \exp(-L^2/L_o^2) \times [\exp(E / \sigma^2) - 1]$

- After about 15 t_{relax}, the process takes off in a runaway **gravothermal catastrophe**
  (An intuitive explanation is tricky -- see B&T-2 7.3.2)
  This "event" is called **core collapse** and leaves a density law $\rho(r) \propto r^{-2.23}$ (infinite at r=0 !)
  since GC are about 20 × t_{relax} old, at least some have probably undergone core collapse
  In practice, core collapse is not as dramatic as its name suggests :
  either
  - the core "runs out of stars" before densities become exotic
  - a hard binary forms which
    → (a) scatters core stars, heating the core and halting core collapse
    → (b) ejects stars from the system, accelerating evaporation
    (the binary acts like a source of nuclear burning in a star --- see below)

- Similar behaviour is found in **stars** : contracting cores heat while expanding envelopes cool
  Lynden-Bell and Eggleton (1980) derive power-law of -9/4 (-2.25) for a conducting gas sphere
  (obviously, this had no nuclear energy source, so could undergo gravothermal collapse)

(ii) Equipartition

- Violent relaxation during formation leaves all stars the same **velocity distribution**
  consequently heavier stars have more kinetic energy
  this is unlike a gas, where molecules have the **same** kinetic energy (heavier ones move slower)

- 2-body encounters mimic molecular interactions: energy passed from high mass to low mass stars
  in the limit of complete interaction, energy is shared equally (hence **equipartition**)

- More massive stars **begin to sink deeper** → **mass segregation**.
  Probably occurred in GCs, though difficult to check since (visible) giants all have similar mass.
  May have played role in galaxy clusters, but other effects (dynamical friction, mergers) confuse interpretation.

(iii) Escape (Ejection and Evaporation)

- Encounters can result in stars with $V > V_{esc}$
  this can occur in two ways :
  a single encounter gives the star sufficient energy to escape (ejection)
  a star slowly wanders into unbound phase space due to many distant encounters
  From the analysis above (10.a.iii), the second is much more important

- Using the fact that $V_{esc}^2 = -2\Phi(r)$, it is easy to show (B&T p 490) that $<V_{esc}^2> = 4 <V^2>$
  so the rms escape velocity is just twice the rms velocity
  for a Maxwellian, the fraction with $V > 2V_{rms} = 7 \times 10^{-3}$
  so this fraction is lost every t_{relax} → $t_{evap} \approx 140\ t_{relax}$
  more detailed calculations confirm this

- The process speeds up in a galaxy tidal field, since V_{esc} is reduced (see [Topic 12.3.b])

- Evaporation + equipartition → **less massive** stars evaporate first (higher velocities)
  explains unusually low M/L (~2) for GCs compared to other pop II objects (M/L ~ 10)

- The observed distribution of t_{relax} for the ~150 MW GCs shows essentially **none** < 10^8 years
  selection effect : since $t_{evap} \sim 100\ t_{relax}$ these GCs have probably already evaporated
  suggests young MW may have had many more GCs

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(11) Further Topics

We defer a few topics of Stellar Dynamics to later sections :

- Dynamical Friction [Topic 12 § 3a]
- Tidal Evaporation [Topic 12 § 3b]
- Slow (adiabatic) & Fast (impulsive) Encounters [Topic 12 § 3c]
- Mergers [Topic 12 § 3d]
- The effects of central black holes on galaxy nuclei [Topic 14 § 5]

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