解： 1） $A_1, A_2, \cdots, A_n$相互独立$\Leftrightarrow \overline{A_1}, \overline{A_2}, \cdots, \overline{A_n}$相互独立

$$P(\overline{A_1} \cdot \overline{A_2} \cdot \cdots \cdot \overline{A_n}) = \prod_{i=1}^{n} P(\overline{A_i}) = \prod_{i=1}^{n} [1 - P(A_i)] = \prod_{i=1}^{n} (1 - p_i)$$

2）

$$P(A_1 + A_2 + \cdots + A_n) = 1 - P(\overline{A_1 + A_2 + \cdots + A_n})$$
$$= 1 - P(\overline{A_1} \cdot \overline{A_2} \cdot \cdots \cdot \overline{A_n})$$
$$= 1 - \prod_{i=1}^{n} (1 - p_i)$$

3）

$$(A_i \bigcap_{j \neq i}^{n} \overline{A_j}) \bigcap (A_k \bigcap_{j \neq k}^{n} \overline{A_j}) = A_i \cap \overline{A_i} \cap A_k \cap \overline{A_k} \bigcap_{j \neq i,k}^{n} \overline{A_j} = \phi$$

$$P(\bigcup_{i=1}^{n} (A_i \bigcap_{j \neq i}^{n} \overline{A_j})) = \sum_{i=1}^{n} P(A_i \bigcap_{j \neq i}^{n} \overline{A_j})$$
$$= \sum_{i=1}^{n} [P(A_i) \prod_{j \neq i}^{n} P(\overline{A_j})]$$
$$= \sum_{i=1}^{n} [p_i \prod_{j \neq i}^{n} (1 - p_j)]$$

证： 1） $P(A\cap B)=P(A-A\cap\overline{B})=P(A)-P(A\cap\overline{B})$

$$\geq P(A)-P(\overline{B})=1-P(\overline{A})-P(\overline{B})$$

2） $P(\bigcap_{i=1}^{\infty}A_i)=1-P(\overline{\bigcap_{i=1}^{\infty}A_i})=1-P(\bigcup_{i=1}^{\infty}\overline{A_i})$

$$\geq 1-\sum_{i=1}^{\infty}P(\overline{A_i})$$

解： 从20个球中取10个球的取法 C_{20}^{10}

这10个球中k个同号的次数 $2\times\mathrm{C}_{10}^{k}\mathrm{C}_{10}^{10-k}$

特等奖 $2/\mathrm{C}_{20}^{10}=1/92378$

一等奖 $2\times\mathrm{C}_{10}^{9}\mathrm{C}_{10}^{1}/\mathrm{C}_{20}^{10}=100/92378$

二等奖 $2\times\mathrm{C}_{10}^{8}\mathrm{C}_{10}^{2}/\mathrm{C}_{20}^{10}=2025/92378$

三等奖 $2\times\mathrm{C}_{10}^{7}\mathrm{C}_{10}^{3}/\mathrm{C}_{20}^{10}=14400/92378$

四等奖 $2\times\mathrm{C}_{10}^{6}\mathrm{C}_{10}^{4}/\mathrm{C}_{20}^{10}=44100/92378$

无奖 $\mathrm{C}_{10}^{5}\mathrm{C}_{10}^{5}/\mathrm{C}_{20}^{10}=31752/92378$

解：$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-1}^{1} \frac{A}{\sqrt{1-x^2}}\,\mathrm{d}x = A\pi = 1$$

$$A = 1/\pi$$

$$P(-1/2 < x < 1/2) = \int_{-1/2}^{1/2} f(x)\,\mathrm{d}x = 1/3$$

解：由题意，有

$$F(X) = P(\xi \le X) \propto \pi X^2$$

令 $$F(X) = C\pi X^2$$

而 $$P(\xi \le 2) = 1$$

故 $$C = \frac{1}{4\pi}$$

$$F(X) = \begin{cases} 0 & X < 0 \\ \frac{1}{4}X^2 & 0 \le X \le 2 \\ 1 & X > 2 \end{cases}$$

解： 1) $P(\xi \le 2) = F(2) = 1 - \mathrm{e}^{-2}$

$P(\xi > 3) = 1 - F(3) = \mathrm{e}^{-3}$

2) $f(x) = \dfrac{\mathrm{d}F(x)}{\mathrm{d}x} = \begin{cases} \mathrm{e}^{-x} & x \ge 0 \\ 0 & x < 0 \end{cases}$

★ 解：飞机被甲、乙、丙击中分别记为A_1、A_2、A_3，则事件A_1、A_2、A_3、$\overline{A_1}$、$\overline{A_2}$、$\overline{A_3}$之间相互独立，且$P(A_1)=0.4, P(A_2)=0.5, P(A_3)=0.7$

飞机被一人击中(B_1)的概率

$$P(B_1)=P(A_1\overline{A_2}\,\overline{A_3})+P(\overline{A_1}A_2\overline{A_3})+P(\overline{A_1}\,\overline{A_2}A_3)$$

$$=P(A_1)P(\overline{A_2})P(\overline{A_3})+P(\overline{A_1})P(A_2)P(\overline{A_3})+P(\overline{A_1})P(\overline{A_2})P(A_3)=0.36$$

飞机被两人击中(B_2)的概率

$$P(B_2)=P(A_1A_2\overline{A_3})+P(A_1\overline{A_2}A_3)+P(\overline{A_1}A_2A_3)$$

$$=P(A_1)P(A_2)P(\overline{A_3})+P(A_1)P(\overline{A_2})P(A_3)+P(\overline{A_1})P(A_2)P(A_3)=0.41$$

飞机被三人击中(B_3)的概率

$$P(B_3)=P(A_1A_2A_3)=P(A_1)P(A_2)P(A_3)=0.14$$

飞机被击落记为C，则被一人、两人、三人击落的概率分别为

$$P(C\mid B_1)=0.2, P(C\mid B_2)=0.6, P(C\mid B_3)=1$$

飞机被击落的概率 $P(C)=\sum_{i=1}^{3}P(B_i)P(C\mid B_i)=0.458$

飞机未被击落而被两人击中的概率

$$P(B_2\mid\overline{C})=P(B_2\overline{C})/P(\overline{C})=[P(B_2)-P(B_2C)]/P(\overline{C})$$

$$=[P(B_2)-P(B_2)P(C\mid B_2)]/[1-P(C)]=0.3026$$

解：记甲、乙两人投中为A、B，则$P(A)=0.6$，$P(B)=0.7$

两人三次投篮投中k次的概率分别为 $C_3^k[P(A)]^k[P(\overline{A})]^{3-k}$

$C_3^k[P(B)]^k[P(\overline{B})]^{3-k}$

两人投中次数相等的概率为

$$\sum_{k=0}^{3}C_3^k[P(A)]^k[P(\overline{A})]^{3-k}\,C_3^k[P(B)]^k[P(\overline{B})]^{3-k}=0.32076$$

甲比乙投中次数多的概率为

$$\sum_{l=0}^{3}\sum_{k=l+1}^{3}C_3^k[P(A)]^k[P(\overline{A})]^{3-k}\,C_3^l[P(B)]^l[P(\overline{B})]^{3-l}=0.243$$

解：$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int_0^{\infty}\int_0^{\infty}C\,\mathrm{e}^{-(3x+4y)}\mathrm{d}x\,\mathrm{d}y=1\Rightarrow C=12$

$$F(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y}f(x,y)\,\mathrm{d}x\,\mathrm{d}y=\begin{cases}(1-\mathrm{e}^{-3x})(1-\mathrm{e}^{-4y}) & x>0,y>0\\ 0 & \text{其它}\end{cases}$$

$$P(0<x\le 1,0<y\le 2)=\int_0^2[\int_0^1 f(x,y)\,\mathrm{d}x]\,\mathrm{d}y=0.9499$$

★ 解： 1) $z \le 0 \quad F(z) = 0$

$$z > 0 \quad F(z) = P(\zeta < z)$$

$$= \iint\limits_{\sqrt{x^2+y^2}<z} 4xy\,\mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^{\pi/2} \mathrm{d}\theta \int_0^z 4r^3 \sin\theta\cos\theta\,\mathrm{e}^{-r^2}\,\mathrm{d}r$$

$$= 1 - \mathrm{e}^{-z^2}(1+z^2)$$

$$f(z) = \begin{cases} 2\mathrm{e}^{-z^2} z^3 & z > 0 \\ 0 & z \le 0 \end{cases}$$

$$E(z) = \int_{-\infty}^{\infty} zf(z)\,\mathrm{d}z = \int_0^{\infty} 2\mathrm{e}^{-z^2} z^4 dz = 3\sqrt{\pi}/4$$

2) $E(z) = \int_{-\infty}^{\infty} zf(z)dz$

$$= \int_0^{\infty}\int_0^{\infty} \sqrt{x^2+y^2} \cdot 4xy\,\mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y$$

$$= 3\sqrt{\pi}/4$$

解： $D(X \pm Y) = E\{[(X \pm Y) - E(X \pm Y)]^2\}$

$$= E\{[X - E(X)]^2\} + E\{[Y - E(Y)]^2\}$$

$$\pm 2E\{[X - E(X)][Y - E(Y)]\}$$

$$= D(X) + D(Y) \pm 2\mathrm{COV}(X, Y)$$

$$= D(X) + D(Y) \pm 2\rho\sqrt{D(X)D(Y)}$$

$$= 25 + 36 \pm 2 \times 0.4 \times \sqrt{25 \times 36}$$

$$= 61 \pm 24$$

$$D(X + Y) = 85$$

$$D(X - Y) = 37$$

解： $f(x)=\begin{cases}1/2 & x\in[2,4]\\ 0 & x\notin[2,4]\end{cases}$

1) $P(-1\le\xi<3)=\int_{-1}^{3}f(x)\,\mathrm{d}\,x=1/2$

2) $P\{(\xi-3)^2<0.25\}=P(2.5<\xi<3.5)=\int_{2.5}^{3.5}f(x)\,\mathrm{d}\,x=1/2$

解： 1) $P(-4<\xi<10)=F(10)-F(-4)=\Phi(\frac{10-3}{2})-\Phi(\frac{-4-3}{2})$

$=2\Phi(3.5)-1=2\times0.9997674-1=0.9995348$

2) $P(|\xi|>2)=1-F(2)+F(-2)=1-\Phi(\frac{2-3}{2})+\Phi(\frac{-2-3}{2})$

$=\Phi(0.5)+1-\Phi(2.5)=0.6915+1-0.993790=0.69771$

3) $P(\xi>C)=1-F(C)$

$=P(\xi\le C)=F(C)$

$F(C)=\Phi(\frac{C-3}{2})=0.5=\Phi(0)$

$C=3$

解： 1) $E(\eta)=E(2\xi)$
$$=2\int_{-\infty}^{\infty} xf(x)\,\mathrm{d}x$$
$$=2\int_{0}^{\infty} x\mathrm{e}^{-x}\,\mathrm{d}x$$
$$=2$$

2) $E(\eta)=E(\mathrm{e}^{-2\xi})$
$$=\int_{-\infty}^{\infty}\mathrm{e}^{-2x}\, f(x)\,\mathrm{d}x$$
$$=\int_{0}^{\infty}\mathrm{e}^{-2x}\mathrm{e}^{-x}\,\mathrm{d}x$$
$$=1/3$$

解： $E(\xi_1)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 f(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2=7/6$

$E(\xi_1)=E(\xi_2)=7/6$

$$\mathrm{COV}(\xi_1,\xi_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1-7/6)(x_2-7/6)f(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2$$
$$=-1/36$$

$$D(\xi_1)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1-7/6)^2 f(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2$$
$$=11/36$$

$D(\xi_2)=D(\xi_1)=11/36$

$\rho(\xi_1,\xi_2)=\mathrm{COV}(\xi_1,\xi_2)/\sqrt{D(\xi_1)D(\xi_2)}=-1/11$

解： $$E(\omega)=E(\xi+\eta+\zeta)$$
$$=E(\xi)+E(\eta)+E(\zeta)$$
$$=1+1-1=1$$
$$D(\omega)=D(\xi+\eta+\zeta)$$
$$=D(\xi)+D(\eta)+D(\zeta)+2E\{[\xi-E(\xi)][\eta-E(\eta)]\}$$
$$+2E\{[\eta-E(\eta)][\zeta-E(\zeta)]\}+2E\{[\xi-E(\xi)][\zeta-E(\zeta)]\}$$
$$=D(\xi)+D(\eta)+D(\zeta)+2\rho(\xi,\eta)\sqrt{D(\xi)D(\eta)}$$
$$+2\rho(\eta,\zeta)\sqrt{D(\eta)D(\zeta)}+2\rho(\xi,\zeta)\sqrt{D(\xi)D(\zeta)}$$
$$=1+1+1+0-1+1=3$$

解： $$f(x)=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}} \quad -\infty<x<\infty$$
$$g(y)=\begin{cases}\dfrac{f(-\sqrt{y})+f(\sqrt{y})}{2\sqrt{y}} & y\ge 0\\ 0 & y<0\end{cases}=\begin{cases}\dfrac{1}{\sqrt{2\pi}}y^{-\frac{1}{2}}\mathrm{e}^{-\frac{y}{2}} & y\ge 0\\ 0 & y<0\end{cases}$$

证： $E(X)=\mu_1 \quad D(X)=\sigma_1^{\ 2}$

$E(Y)=\mu_2 \quad D(X)=\sigma_2^{\ 2}$

$$E(\frac{X+Y}{2})=\frac{1}{2}E(X)+\frac{1}{2}E(Y)=\frac{\mu_1+\mu_2}{2}$$

$$D(\frac{X+Y}{2})=(\frac{1}{2})^2D(X)+(\frac{1}{2})^2D(Y)=\frac{\sigma_1^{\ 2}+\sigma_2^{\ 2}}{4}$$

$$\therefore \quad \frac{X+Y}{2}\sim N(\frac{\mu_1+\mu_2}{2},\frac{\sigma_1^{\ 2}+\sigma_2^{\ 2}}{4})$$

$\because$ x,y相互独立 $\therefore$ $f(x,y)=f(x)f(y)$

令 $z=\dfrac{x+y}{2}$

$$f(z)=2\int_{-\infty}^{\infty}f(2z-y,y)\,\mathrm{d}\,y$$

$$=\frac{1}{\sqrt{2\pi\frac{(\sigma_1^{\ 2}+\sigma_2^{\ 2})}{4}}}\exp[-\frac{(z-\frac{\mu_1+\mu_2}{2})^2}{2\frac{(\sigma_1^{\ 2}+\sigma_2^{\ 2})}{4}}]$$

$$\therefore \quad \frac{X+Y}{2}\sim N(\frac{\mu_1+\mu_2}{2},\frac{\sigma_1^{\ 2}+\sigma_2^{\ 2}}{4})$$

解： 设6000粒种子中良种数为ξ，$\xi \sim B(6000,1/6)$

其所占比例与1/6相差不超过ε。由题意，有

$$E(\xi)=1000 \quad D(\xi)=5000/6 \quad P(|\xi/6000-1/6|<\varepsilon)=0.99$$

根据棣莫佛－拉普拉斯定理，有

$$P(|\xi/6000-1/6|<\varepsilon) = P\left(\left|\frac{\xi-1000}{\sqrt{5000/6}}\right| < \frac{6000\varepsilon}{\sqrt{5000/6}}\right)$$

$$=0.99=\Phi(2.5757)-\Phi(-2.5757)$$

即 $$\frac{6000\varepsilon}{\sqrt{5000/6}}=2.5757 \quad \Rightarrow \begin{cases} \varepsilon=0.0124 \\ 6000\varepsilon=75 \end{cases}$$

$$925<\xi<1075$$

解：$S_1^2 = 6.38 \quad S_2^2 = 5.15$

$n_1 = 25 \quad n_2 = 15$

$\because \quad \dfrac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F(n_1 - 1, n_2 - 1)$

$\therefore \quad \sigma_1^2/\sigma_2^2$的置信区间为$\left(\dfrac{S_1^2}{S_2^2}\dfrac{1}{F_{\alpha/2}}, \dfrac{S_1^2}{S_2^2}\dfrac{1}{F_{1-\alpha/2}}\right)$

$\because \quad 1-\alpha = 90\%$

$\therefore \quad \alpha = 0.1$

查F分布表得到

$F_{0.05}(24,14) = 2.35$

$F_{0.95}(24,14) = 1/F_{0.05}(14,24) = 1/2.13$

将S_1^2、S_2^2代入，得到

σ_1^2/σ_2^2的置信区间为(0.527,2.64)

解： 1) $L(\theta)=\theta^n\prod_{i=1}^{n}x_i^{\theta-1}$

$$\ln L(\theta)=n\ln\theta+(\theta-1)\sum_{i=1}^{n}\ln x_i$$

$$\frac{\partial \ln L(\theta)}{\partial\theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\ln x_i=0$$

$$\hat{\theta}=-n\Big/\sum_{i=1}^{n}\ln x_i$$

2) $L(\theta)=\theta^n\alpha^n\prod_{i=1}^{n}x_i^{\alpha-1}\exp(-\sum_{i=1}^{n}\theta x_i^{\alpha})$

$$\ln L(\theta)=n\ln\theta+n\ln\alpha+(\alpha-1)\sum_{i=1}^{n}\ln x_i-\sum_{i=1}^{n}\theta x_i^{\alpha}$$

$$\frac{\partial \ln L(\theta)}{\partial\theta}=\frac{n}{\theta}-\sum_{i=1}^{n}x_i^{\alpha}=0$$

$$\hat{\theta}=n\Big/\sum_{i=1}^{n}x_i^{\alpha}$$

解：$\sigma^2 = 0.09 \quad \Rightarrow \quad \sigma = 0.3$

$\overline{X} = (12.6 + 13.4 + 12.8 + 13.2)/4 = 13$

$\because \quad \dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$

$\therefore \quad \mu$的置信区间为$(\overline{X} - u_\alpha \dfrac{\sigma}{\sqrt{n}}, \overline{X} + u_\alpha \dfrac{\sigma}{\sqrt{n}})$

$\because \quad 1 - \alpha = 95\% \quad \Rightarrow \quad \alpha = 0.05$

$\therefore \quad u_\alpha = 1.959964$

将$\overline{X}$、σ、n、u_α代入，得到

μ的置信区间为$(12.706, 13.294)$

解： 若 $Y = c[(X_1+X_2+X_3)^2+(X_4+X_5+X_6)^2] \sim \chi^2(n)$

则 $Z_1 = \sqrt{c}(X_1+X_2+X_3) \sim N(0,1)$

$Z_2 = \sqrt{c}(X_4+X_5+X_6) \sim N(0,1)$

而 $X_1, X_2, \cdots, X_6 \sim N(0,1)$

有 $c+c+c=1$

故 $c = 1/3$ 且 $n=2$

解： $\because$ $X \sim N(3.4, 6^2)$

$\therefore$ $\overline{X} \sim N(3.4, 6^2/n)$

而 $P(1.4 < \overline{X} < 5.4) = P(|\overline{X} - 3.4| < 2)$

$$= P\left(\left|\frac{\overline{X}-3.4}{6/\sqrt{n}}\right| < \frac{2}{6/\sqrt{n}}\right) = \Phi(\frac{\sqrt{n}}{3}) - \Phi(\frac{\sqrt{n}}{3})$$

$$= 2\Phi(\frac{\sqrt{n}}{3}) - 1 \geq 0.95$$

$\therefore$ $\frac{\sqrt{n}}{3} \geq 1.96$

$n \geq 35$

★ 证： $\because \quad X_1, X_2, \cdots, X_n \sim N(\mu, \sigma^2)$

$\therefore \quad Y_1 = (X_1 + X_2 + \cdots + X_6)/6 \sim N(\mu, \sigma^2/6)$

$Y_2 = (X_7 + X_8 + X_9)/3 \sim N(\mu, \sigma^2/3)$

$\therefore \quad Y_1 - Y_2 \sim N(0, \sigma^2/2)$

$\therefore \quad \dfrac{\sqrt{6}(Y_1 - Y_2)}{\sigma} \sim N(0,1)$

而 $X_7, X_8, X_9 \sim N(\mu, \sigma^2)$

$\therefore \quad 2S^2/\sigma^2 \sim \chi^2(2)$

$\therefore \quad Z = \dfrac{\sqrt{2}(Y_1 - Y_2)/\sigma}{\sqrt{2S^2/\sigma^2/2}} = \dfrac{\sqrt{2}(Y_1 - Y_2)}{S} \sim t(2)$

解： 1. $P(|\overline{X}-\mu|<2)=P\left(\left|\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right|<\frac{2}{\sigma/\sqrt{n}}\right)$

$=2\Phi(1.6)-1=0.8904$

2. $P(|\overline{X}-\mu|<2)=P\left(\left|\frac{\overline{X}-\mu}{S/\sqrt{n}}\right|<\frac{2}{S/\sqrt{n}}\right)$

$=P\left(t_{-1.754}(15)<\frac{\overline{X}-\mu}{S/\sqrt{n}}<t_{1.754}(15)\right)=0.90$

3. $P(\frac{S^2}{\sigma^2}\leq 2.041)=P(\chi^2(15)\leq 30.615)$

$=1-P(\chi^2(15)>30.615)=0.99$

$\because$ $S^2=\chi^2(n-1)\sigma^2/(n-1)$

$\therefore$ $D(S^2)=\sigma^4/15^2\cdot D[\chi^2(n-1)]=2\sigma^4/15$